Games of Elimination

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September 2010
HSE

Competition

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- Our target: model interactions with actions that may be directed at a specific opponent.
  Examples: political games, mafia control, negative ads, litigation, patent races, industrial espionage, ...

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- We analyze games with 2 and 3+ players.
- Cooperation (!) may arise even in these games — in the face of death.

The model

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- game ends when EITHER no more than 1 "alive" player is left OR when all shot in the air.
- Payoffs: if K players are "alive" at the end, each receives X_K, others 0.
  $Y = X_1 > X_2 > \cdots > X_N$. For $i > 1$, $X_i < Y/i$.

The duel

Player i's payoff

$$
\begin{aligned}
Z_i &= (1-\alpha_j)\alpha_i Y + (1-\alpha_j)(1-\alpha_i)Z_i; \\
Z_i &= \frac{\alpha_i - \alpha_i\alpha_j}{\alpha_i + \alpha_j - \alpha_i\alpha_j} Y. \qquad (1)
\end{aligned}
$$

In the case $\alpha_i = \alpha_j = \alpha$ the payoff is $Z = \frac{1-\alpha}{2-\alpha} Y$.

Note that Z_i is the payoff that player i can guarantee to herself no matter what is the strategy of the opponent.

The duel

Lemma

"Peace," that is simultaneous shooting in the air, cannot be sustained in equilibrium.

$$D_i + D_j = \frac{\alpha_i + \alpha_j}{\alpha_i + \alpha_j - \alpha_i \alpha_j} Y > Y > X_2 + X_2.$$

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- Note, that if $X_2 > Y/2$, peace can be sustained, but not with "strong" players.

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Lemma

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$$A_{i1} = \alpha_i Y + (1-\alpha_j)(1-\alpha_i)A_i, \quad (2)$$
$$A_{j2} = (1-\alpha_i)\alpha_j Y + (1-\alpha_j)(1-\alpha_i)A_j. \quad (3)$$

Note that for all i, $A_{i2} = Z_i$, and so $A_{i1} = Y - Z_j$.

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  Note that for all i, $A_{i2} = Z_i$, and so $A_{i1} = Y - Z_j$.
- By shooting in sequence, the players eliminate the undesirable event of both of them dying.
- This may explain why some duels have "sequential" rules.

Other equilibria

Lemma

The set of pure SPE is completely described by the pair (T, k), *where* $T \in \mathbb{N} \cup \{\infty\}$ *is the period in which "polite war" starts, and* k *is the player whose turn is to shoot first. Before period* T *the players follow the strategies of the "war" equilibrium.*

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- Mixed equilibria: ONE player mixes and can trigger the "polite war."

Truels

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- **Thm:** The one-shot deviation principle.

Stationary equilibria

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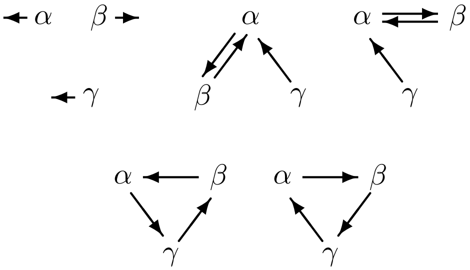
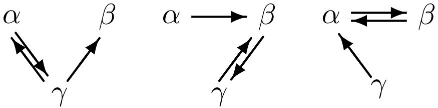
- The following pure stationary SPE may exist (under certain conditions on α, β, γ).



Non-stationary equilibria

"Efficient war"



Various types of "polite war" also exist.

Peace sustainable?

- $X_2 = X_3 = 0$: pure truel.
  Peace cannot be sustained.
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- $X_2 = X_3 = 0$: pure truel.
  Peace cannot be sustained.
  Never optimal to "abstain" in stationary equilibria.
- $X_3 > 0$.
  Peace can be sustained.
  Easier for strong players (!).
  Non-monotone condition (!).

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Three kinds of cooperation may emerge

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Cooperation

Three kinds of cooperation may emerge

1. Peace: payoffs to peace have to be sufficiently high. Harder to sustain with strong players when $N = 2$, easier when $N > 2$.
2. "Polite" and "efficient war": players eliminate a "no one survives" event.
3. Cycles: players select different targets to avoid duplication of effort.

Conclusions and Future directions

- Interesting equilibrium patterns.
- Various kinds of cooperation emerge.
- For $N = 3$, peace is hardest to achieve for "intermediate" shooters.
- "Weak" players may have largest payoffs, and benefit from their opponents getting stronger.

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- Costly actions/ abilities.
- Experiments.